Existence of Periodic Limiting Regimes for a Nonlinear System of Ordinary Differential Equations

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In this paper certain results of the author (3),[1] obtained for the equation $y' = f(x,y)$ are generalized to a system of nonlinear ordinary differential equations. In particular, we prove the existence under certain conditions of a periodic solution of the system and also asymptotic convergence of other solutions to the periodic solution. For the system

$$dX/dt = F(X) + G(t)$$

these results have been obtained by B. P. Demidovich (1), whose results were generalizations of well-known results of N. N. Luzin (2).

Let us consider the nonlinear system

$$dX/dt = F(X,t)$$

where X and $F(X,t)$ are real column vectors:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \qquad F(X,t) = \begin{pmatrix} f_1(X,t) \\ f_2(X,t) \\ \cdot \\ \cdot \\ \cdot \\ f_n(X,t) \end{pmatrix}$$

in Euclidean space. $F(X,t)$ is continuously differentiable in $x_1, \ldots, x_n$, and is periodic in t with period T $(T > 0)$

$$F(X,t + T) \equiv F(X,t) \qquad [2]$$

Definition 1. A symmetric matrix $A = [a_{ij}]$, $(a_{ij} = a_{ji})$ is called sign-definite if all its eigenvalues have the same sign. It is positive-definite if all its eigenvalues are positive, and negative-definite if all its eigenvalues are negative.

Definition 2. Let X be a vector variable. The symmetric matrix $A(X) = [a_{ij}(X)]$ is called uniformly sign-definite if all its eigenvalues have the same sign and are bounded away from zero. It is uniformly positive-definite if the lower bound of the eigenvalues is positive, and is uniformly negative-definite if the upper bound of the eigenvalues is negative.

Lemma: Let $X = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$ be a column vector, and let $A(X) = [a_{ij}(X)]$ be a square, symmetric, uniformly positive-definite matrix.

In this case the scalar product

$$[A(X)X,X] = \sum_{i,j=1}^{n} a_{ij}(X)\, x_i x_j \qquad [3]$$

is a positive-definite function, and we have

$$[A(X)X,X] \geqq h||X||^2 \qquad [4]$$

where h is a positive constant, and

$$||X|| = \sqrt{\sum_{i=1}^{n} x_i^2} \qquad [5]$$

is the norm of the vector X.

Proof: Let us consider the auxiliary quadratic form

$$[A(X)Y,Y] = \sum_{i,j=1}^{n} a_{ij}(X) y_i y_j \qquad [6]$$

in the variable

$$Y = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix}$$

where the variable X plays the role of a parameter. Since the matrix $A(X)$ is symmetric for any fixed value of the vector X, there exists an orthogonal transformation

$$Y = U(X)Z \qquad [7]$$

where

$$Z = \begin{pmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{pmatrix}$$

which reduces the quadratic form [6] to canonical form

$$[A(X)Y,Y] = \sum_{i=1}^{n} \lambda_i(X)\, z_i^2 \qquad [8]$$

By virtue of the uniform positive-definiteness of the matrix $A(X)$, we have

$$\inf \lambda_i(X) \geqq h \qquad (i = 1, \ldots, n)$$

where h is a positive number. Therefore

$$[A(X)Y,Y] \geqq h||Z||^2$$

Since the transformation $U(X)$ is orthogonal

$$||Z|| = ||Y||$$

so that

$$[A(X)Y,Y] \geqq h||Y||^2 \qquad [9]$$

Setting $Y = X$ in the inequality [9], we finally obtain

$$[A(X)X,X] \geqq h||X||^2$$

This completes the proof of the lemma.

Similarly, if the matrix $A(X)$ is uniformly negative-definite, we have the inequality

$$[A(X),X] \leqq -h||X||^2$$

Existence of a Periodic Solution

Let

$$X = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \text{ and } F(X,t) = \begin{pmatrix} f_1(X,t) \\ \cdot \\ \cdot \\ \cdot \\ f_n(X,t) \end{pmatrix}$$

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[1] Numbers in parentheses indicate References at end of paper.

where the $f_i(X,t)$ are continuously differentiable functions of the variables $x_1, \ldots, x_n$ and are continuous and periodic in t. Let

$$\omega_i = \begin{pmatrix} \omega_{i1} \\ \cdot \\ \cdot \\ \cdot \\ \omega_{in} \end{pmatrix} \qquad (i = 1, \ldots, n)$$

denote a set of independent vector variables. Let us form the matrix

$$\overline{W}(\omega_1, \ldots, \omega_n,t) = \left[\frac{\partial f_i}{\partial x_j}(\omega_i,t)\right] \tag{10}$$

and let us symmetrize it, setting

$$\overline{W}_s(\omega_1, \ldots, \omega_n,t) = \tfrac{1}{2}[\overline{W}(\omega_1, \ldots, \omega_n,t) + \overline{W}'(\omega_1, \ldots, \omega_n,t)] \tag{11}$$

where $\overline{W}'(\omega_1, \ldots, \omega_n,t)$ is the transpose of the matrix $\overline{W}(\omega_1, \ldots, \omega_n,t)$. The matrix $\overline{W}(\omega_1, \ldots, \omega_n,t)$ is called the generalized Jacobian matrix of the system of functions $F(X,t)$, and the matrix $\overline{W}_s(\omega_1, \ldots, \omega_n,t)$ is called the symmetrized generalized Jacobian matrix of the system of functions $F(X,t)$.

Theorem: If the symmetrized generalized Jacobian matrix [11] *of the system of functions $F(X,t)$ is a uniformly sign-definite matrix, then system* [1] *has a periodic solution of period T.*

Proof: The conditions imposed on the vector $F(X,t)$ imply for each t_0 and X_0 the existence of a unique solution of system [1]:

$$X = X(t,t_0,X_0)$$

such that

$$X(t_0,t_0, X_0) = X_0$$

It is easy to show that

$$\frac{1}{2}\frac{d}{dt}\{\|X\|^2\} = \left(\frac{dX}{dt}, X\right)$$

from which we have, using [1]:

$$\frac{1}{2}\frac{d}{dt}\{\|X\|^2\} = [F(X,t),X] = \{[F(X,t) - F(0,t)],X\} + [F(0,t),X] \tag{12}$$

Now applying the mean-value theorem to the difference $F(X,t) - F(0,t)$, we obtain

$$\begin{bmatrix} f_1(x_1,x_2, \ldots, x_n,t) - f_1(0,0, \ldots, 0,t) \\ f_2(x_1,x_2, \ldots, x_n,t) - f_2(0,0, \ldots, 0,t) \\ \cdots \\ f_n(x_1,x_2, \ldots, x_n,t) - f_n(0,0, \ldots, 0,t) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\theta_1x_1,\theta_1x_2, \ldots \theta_1x_n,t)x_1 + \frac{\partial f_1}{\partial x_2}(\theta_1x_1,\theta_1x_2, \ldots, \theta_1x_n,t)x_2 \\ + \ldots + \frac{\partial f_1}{\partial x_n}(\theta_1x_1,\theta_1x_2, \ldots, \theta_1x_n,t)x_n \\ \frac{\partial f_2}{\partial x_1}(\theta_2x_1,\theta_2x_2, \ldots, \theta_2x_n,t)x_1 + \frac{\partial f_2}{\partial x_2}(\theta_2x_1,\theta_2x_2, \ldots, \theta_2x_n,t)x_2 \\ + \ldots + \frac{\partial f_2}{\partial x_n}(\theta_2x_1, \theta_2x_2, \ldots, \theta_2x_n,t)x_n \\ \cdots \\ \frac{\partial f_n}{\partial x_1}(\theta_nx_1,\theta_nx_2, \ldots, \theta_nx_n,t)x_1 + \frac{\partial f_n}{\partial x_2}(\theta_nx_1,\theta_nx_2, \ldots, \theta_nx_n,t)x_2 \\ + \ldots + \frac{\partial f_n}{\partial x_n}(\theta_nx_1,\theta_nx_2, \ldots, \theta_nx_n,t)x_n \end{bmatrix}$$

where $0 < \theta_i = \theta_i(X) < 1$.

If we now set the matrix

$$\left[\frac{\partial f_i}{\partial x_j}(\theta_iX,t)\right] = W(X,t)$$

we have

$$F(X,t) - F(0,t) = W(X,t)X$$

Hence

$$\frac{1}{2}\frac{d}{dt}\{\|X\|^2\} = [W(X,t)X,X] + [F(0,t),X] \tag{13}$$

For definiteness let the symmetrized generalized Jacobian matrix

$$\overline{W}_s(\omega_1, \ldots, \omega_n,t) = \left\{\frac{1}{2}\left[\frac{\partial f_i}{\partial x_j}(\omega_i,t) + \frac{\partial f_j}{\partial x_i}(\omega_j,t)\right]\right\}$$

be uniformly negative-definite.

Let

$$W_s(X,t) = \tfrac{1}{2}[W(X,t) + W'(X,t)]$$

be the matrix obtained by symmetrizing $W(X,t)$. Then clearly

$$[W(X,t)X,X] = [W_s(X,t)X,X]$$

Since

$$W_s(X,t) = \overline{W}_s(\theta_1X, \ldots, \theta_nX,t)$$

the matrix $W_s(X,t)$ is also uniformly negative-definite. Hence by virtue of the lemma we have

$$[W(X,t)X,X] \leqq -h\|X\|^2 \tag{14}$$

where h is a positive number.

Moreover, assuming that $\|X\| \neq 0$, and taking into account the boundedness of $F(0,t)$, we obtain

$$\left|\left(F(0,t), \frac{X}{\|X\|}\right)\right| < C \tag{15}$$

where C is a constant.

Hence by virtue of [12] we have

$$(d/dt)(\|X\|) < -h\|X\| + C \tag{16}$$

if $\|X\| \neq 0$.

Let us consider the sphere

$$\|X\| = R \tag{17}$$

where

$$R > C/h \qquad [18]$$

It follows from formula [16] that every solution $X = X(t,t_0,X_0)$ of Eq. [1] which originates for $t = t_0$ at an arbitrary point of the closed ball $\bar{V}_R = \{\|X\| \leqq R\}$ must remain within this ball as $t \to +\infty$ and hence can be continued for all t:

$$t_0 \leqq t < +\infty$$

Let t_0 be fixed, and let

$$X_1 = X(t_0 + T, t_0, X_0) \qquad [19]$$

where T is the period of $F(X,t)$.

Let S denote the transformation of the space E_n which sends the point X_0 into the point X_1:

$$X_1 = S(X_0) \qquad [20]$$

The transformation S, by virtue of the uniqueness theorem for solutions of differential equations and the periodicity in t with period T of the right member of Eq. [1], is one to one, and maps the closed ball $\bar{V}_R$ into a proper subset of it. Indeed, if there were a point $Y_0 \epsilon \bar{V}_R$ such that $Y_1 = S(Y_0) \bar{\epsilon} V_R$, where V_R is the interior of the ball $\bar{V}_R$, then for the solution $X = X(t,t_0,Y_0)$ there would exist a time $\bar{t}$, $t_0 \leqq \bar{t} \leqq t_0 + T$, such that

$$\|X(\bar{t},t_0,Y_0)\| = R$$

and

$$(d/dt)\{\|X(\bar{t},t_0,Y_0)\|\} \geqq 0$$

which is impossible by virtue of the inequalities [16] and [18]. Hence by the well-known Brouwer fixed-point theorem (4) we conclude that there exists a point $X^* \epsilon V_R$ which is invariant under the transformation S. Clearly the solution

$$X^*(t) = X(t,t_0,X^*)$$

is periodic with period T. This proves the theorem.

Theorem: Under the same conditions all solutions of system [1] *converge as* $t \to +\infty$ *(or as* $t \to -\infty$*) to the periodic solution of the system, i.e., system* [1] *has a periodic limiting regime.*

Proof: Let $X = X(t)$ be an arbitrary solution of the system [1], and let $X^* = X^*(t)$ be the periodic solution of this system. We have

$$dX/dt = F(X,t) \quad \text{and} \quad dX^*/dt = F(X^*,t)$$

Thus

$$(d/dt)(X - X^*) = F(X,t) - F(X^*,t)$$

and hence

$$\frac{1}{2}\frac{d}{dt}\{\|X - X^*\|^2\} = \{[F(X,t) - F(X^*,t)], X - X^*\}$$

Now apply the mean-value theorem and obtain

$$F(X,t) - F(X^*,t) = W(X,X^*,t)\,(X - X^*)$$

where

$$W(X,X^*,t) = \left\{\frac{\partial f_i}{\partial x_j}[X^* + \theta_i(X - X^*)]\right\}$$

with $0 < \theta_i < 1$ for $i = 1,2,\ldots,n$. We thus have

$$\frac{1}{2}\frac{d}{dt}\{\|X - X^*\|^2\} = [W(X,X^*,t)(X - X^*),(X - X^*)]$$

Let $\bar{W}_s(\omega_1,\ldots,\omega_n,t)$ be uniformly negative-definite. Now since

$$W(X,X^*,t) = \bar{W}[X^* + \theta_1(X - X^*),\ X^* + \theta_2(X - X^*),\ \ldots,\ X^* + \theta_n(X - X^*),\ t]$$

we obtain on symmetrizing

$$W_s(X,X^*,t) = \tfrac{1}{2}\,W(X,X^*,t) + W'(X,X^*,t) = \bar{W}_s[X^* + \theta_1(X - X^*),\ X^* + \theta_2(X - X^*),\ \ldots,\ X^* + \theta_n(X - X^*),\ t]$$

so that $W_s(X,X^*,t)$ is also uniformly negative-definite. Since

$$[W(X,X^*,t)(X - X^*),(X - X^*)] = [W_s(X,X^*,t)(X - X^*),(X - X^*)]$$

we have by virtue of the lemma

$$[W(X,X^*,t)(X - X^*),(X - X^*)] \leqq -h\|X - X^*\|^2$$

where h is a positive constant.

Consequently

$$\frac{1}{2}\frac{d}{dt}\{\|X - X^*\|^2\} \leqq -h\|X - X^*\|^2$$

or

$$\frac{d}{dt}\,[e^{2h(t-t_0)}\|X(t) - X^*(t)\|^2] \leqq 0$$

Hence for $t \geqq t_0$ we have

$$\|X(t) - X^*(t)\| \leqq \|X(t_0) - X^*(t_0)\|e^{-h(t-t_0)}$$

and

$$\lim_{t\to+\infty} \|X(t) - X^*(t)\| = 0$$

which proves the theorem.

References

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2 Luzin, N. N., "On the qualitative investigation of the equations of motion of trains," Matematicheskii Sbornik (Mathematical Symposium) **39**, no. 3, 1032.

3 Samedova, S. A., "Criteria for the existence and uniqueness of periodic solutions of the equation $y' = f(x,y)$," Trudy Inst. Fiziki i Matematiki, Acad. Nauk Azerbaidzhanskoi SSR (Trans. Inst. Phys. and Math., Acad. Sci. Azerbaidzhan SSR), **IV** (1953).

4 Brouwer: "Über abbildung von mannigfaltigkeiten," Annalen Mathematische **71** (1912).

Reviewer's Comment

The criterion of sign-definiteness of the symmetric part of the Jacobian has been known for autonomous systems. A reference in English is a paper by L. Markus and H. Yamabe,[2] the references of which are largely to Russian literature. The most prominent name in this area is Krasovskii, with the work of Zubov and Erugin also being of importance.

The paper under discussion is a generalization of the ideas of the forementioned to a nonautonomous, periodically forced system. The generalization is not a deep one. To the reviewer's knowledge, however, no treatment using this criterion for existence and stability, and particularly for global stability, of periodic solutions has appeared before. This fact, along with the very clear presentation, makes the paper of value. The way in which the hypotheses imply the conclusions is simply and clearly demonstrated.

The criterion presented, although simple and elegant, excludes many important cases. The important case that the

[2] Markus, L. and Yamabe, H., "Global stability criteria for differential systems," Osaka Math. J. 12, 305–317 (1960).

Jacobian has all eigenvalues with negative real part, even though the symmetric part is not negative-definite, is not even considered. (See Markus and Yamabe[2] for some results in this direction.) One problem in which the criterion verifies known results is that of how much damping is needed to insure stability in a problem such as the following. Consider

$$x'' + cx' + a(t)x = 0$$

where $a(t)$ is, say, periodic, positive, and bounded away from zero. The system can be unstable for $c = 0$ and for c small and positive. However, if c is large enough, the system is stable. The criterion of this paper will yield information on a problem of this nature.

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Conservation of the Form of the Maxwellian Distribution in a Relaxing Gas

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This paper investigates the establishment of Maxwellian equilib.ium in a light gas surrounded by an atmosphere of heavy gas. It is assumed that the interaction forces are inversely proportional to the fifth power of the distance. It is shown that if the initial distribution of the light gas is Maxwellian, at a temperature which differs from that of the heavy gas, the form of Maxwellian distribution in the light gas is conserved in the passage to equilibrium, and only the temperature changes. The extension of this result to other interaction laws is briefly considered.

IN THE relaxation of a gas to equilibrium the distribution function changes its form. If the initial distribution function is in equilibrium it may not be disturbed by relaxation under certain conditions. Thus in a system of harmonic oscillators dispersed in a thermostat, if the initial distribution of the oscillators over the vibrational levels is a Boltzman distribution (at a temperature differing from the temperature of the thermostat), the vibrational relaxation occurs in such a way that the distribution function of the oscillators remains strictly Boltzmannian, and the vibrational temperature alone changes (1).[1] The present paper examines another similar example—the conservation of the form of Maxwellian distribution in the process of relaxation.

Let us examine a system consisting of a large number of heavy atoms (of mass M) and a small number of light particles or electrons (of mass m). For simplicity's sake we shall speak henceforth of electrons alone. At the initial moment of time the temperature of the heavy particles (of the thermostat) is equal to T, and the temperature of the electrons is equal to T_0. We assume that the temperature of the thermostat remains constant during relaxation. We select the temperatures T and T_0 in a way that enables us to disregard all inelastic processes connected with the formation of negative and positive ions. Then the establishment of equilibrium in the electron gas will occur only as a result of energy exchange through elastic collisions of the electrons with the atoms of the thermostat.

The behavior of the electron distribution function $f(v,t)$ in the process of relaxation is described by the Boltzmann integral-differential equation. If we consider only elastic electron-atom collisions and assume that the electrons are distributed isotropically in velocity space, then, as shown in Ref. 2, the integral-differential equation becomes a differential equation of the Fokker-Planck type, having the form

$$\frac{\partial f}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}\left\{\frac{kT}{M}\frac{v^3}{\lambda(v)}\frac{\partial f}{\partial v} + \frac{mv^4}{M\lambda(v)}f\right\} \qquad [1]$$

Translated from Vestnik Moskovskogo Universiteta (Bulletin of Moscow University), no. 1, 13–15 (1961). Translated by Primary Sources, New York.

[1] Numbers in parentheses indicate References at end of paper.

where λ is the length of the electron free path. Thus, as has also been done in Ref. 3, we assume that the effective electron-atom collision cross section is inversely proportional to the electron velocity, that is:

$$\lambda = \lambda_0 \frac{v}{V} \qquad \text{where} \qquad V^2 = \frac{kT}{m}$$

This assumption is justified only for the Maxwellian law which assumes an interaction inversely proportional to the fifth power of the distance.

At the initial moment of time, the distribution function looks like

$$f(v,0) = N\left(\frac{m}{2\pi kT_0}\right)^{3/2} \exp\left[-\frac{mv^2}{2kT_0}\right] \qquad [2]$$

We solve Eq. [1] as follows:

$$f(v,t) = N\left(\frac{m}{2\pi k\vartheta(t)}\right)^{3/2} \exp\left[-\frac{mv^2}{2k\vartheta(t)}\right] \qquad [3]$$

where $\vartheta(t)$ is the unknown function determined by the initial condition

$$\vartheta(0) = T_0 \qquad [4]$$

Substituting [3] into [1] and abbreviating somewhat, we obtain for $\vartheta(t)$ the usual relaxation equation

$$\frac{d\vartheta}{dt} = -\frac{1}{\alpha}(\vartheta - T) \qquad [5]$$

the solution of which is

$$\vartheta - T = e^{-(t/\alpha)}(T_0 - T) \qquad [6]$$

where $\alpha = \lambda_0 M/2mV$ is the relaxation time. For an electron gas with $\lambda_0 = 6.5\cdot10^{-3}$ cm, located in a thermostat at a temperature $3kT/2 = 0.03$ ev and $M = 30(1832)m$, $\alpha = 4.06\cdot10^{-5}$ sec (3).

Of course, the field of application of the Maxwellian interaction potential is limited. Here the question arises of the conditions which, when used together with the interaction potential, will allow the form of the Maxwellian distribution